

Solitons and black holes in Einstein-Born-Infeld-dilaton theory

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Abstract

Static spherically symmetric asymptotically flat solutions to the $U(1)$ Born-Infeld theory coupled to gravity and to a dilaton are investigated. The dilaton enters in such a way that the theory is $SL(2, R)$ -symmetric for a unit value of the dilaton coupling constant. We find globally regular solutions for any value of the effective coupling which is the ratio of the gravitational and dilaton couplings; they form a continuous sequence labeled by the sole free parameter of the local solution near the origin. The allowed values of this parameter are bounded from below, and, as the limiting value is approached, the mass and the dilaton charge rise indefinitely and tend to a strict equality (in suitable units). Together with the electric charge they saturate the BPS bound of the corresponding linear $U(1)$ theory. Beyond this boundary the solutions become compact (singular), while the limiting solution at the exact boundary value is non-compact and non-asymptotically flat. The black holes in this theory exist for any value of the event horizon radius and also form a sequence labelled by a continuously varying parameter restricted to a finite interval. The singularity inside the black hole exhibits a power-law mass inflation.

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1 Introduction

Born-Infeld (BI) type [1] generalisations of both Abelian and non-Abelian gauge theories have recently attracted much attention in the context of superstring theory. An Abelian BI action was obtained as the disc open string effective action in an external constant vector field [2, 3] (for references and a recent review see [4]). The computation is valid both for the bosonic string and the superstring and is exact in terms of the sigma-model coupling α' . Alternatively, the BI action was derived as an effective action governing the dynamics of vector fields on D -branes [5]. The BI lagrangian introduces a natural bound for the field strength – a Born-Infeld ‘critical field’ – which should now be regarded as originating from the non-locality of the underlying fundamental theory. A direct consequence of this is the smoothing of the pointlike singularities of the vector field. As was shown already in the original papers by Born and Infeld [1], the Coulomb field of a point charge has a finite energy in this theory.

It is expected that a similar regularization of gravitational singularities should follow from the non-locality of closed strings [6]. However no explicit closed string effective action summing up all terms in α' was obtained so far. A somewhat simpler question is how the singularity of the charged black hole gets modified when gravity is still treated classically, while the dynamics of the vector field is governed by the strings. The guess is that the singularity, although not eliminated, will be smoothed somehow. As was shown by Gibbons and Rasheed [7, 8], the Einstein-Born-Infeld charged black holes are less singular indeed as compared with their Einstein-Maxwell counterpart, the Reissner-Nordström (RN) solution. Namely, there is no RN-type divergent term in the metric near the singularity $g_{00} \sim Q^2/r^2$, but only a Schwarzschild type term $g_{00} \sim m_0/r$. But, contrary to the original Schwarzschild solution, both signs of m_0 are possible now for different values of the electric charge, thus both timelike and spacelike singularities may appear. In the latter case an internal Cauchy horizon is present as in the RN metric.

In fact, within the context of string theory two types of charged black holes are known: besides the RN one there is also a dilatonic black hole [9, 10, 11] with a rather different internal structure. The extremal dilatonic black holes are particularly different from the RN ones: the horizon is moved to the singularity which is therefore lightlike. Moreover, in the string frame the metric associated with the extremal magnetic dilaton black hole turns out to be perfectly regular. Thus the dilaton is also able to produce a regularizing effect on gravitational singularities. A natural question arises concerning the effect of both the BI non-linearity and the dilaton, this is the main subject of the present paper. Since already the flat space BI action gives rise to regular particle-like solutions, we also expect such solutions in the Born-Infeld-dilaton theory and its gravitating generalization. As we will see, there is a one-parameter family of such solutions which, for a limiting value of the parameter, tend (after suitable coordinate rescaling) to the near horizon limit of the extremal dilatonic black hole. The limiting solution saturates the BPS bound. Another class of solutions is black holes. We show that a generic Einstein-Born-Infeld-dilaton black hole has a timelike singularity with a power-low behavior of the local mass function.

The choice of the lagrangian is worth to be discussed in detail. We adopt the $SL(2, R)$ invariant version of the dilaton-axion coupled Born-Infeld action [12, 8, 13], which is a direct Born-Infeld type generalisation of the toroidally compactified heterotic string effective theory ($N = 4$, $D = 4$ supergravity). The main reason is that we would like to make contact with the dilatonic black holes [9, 11]. This version of the theory is

also distinguished as the unique dilaton generalization of the original Born-Infeld theory exhibiting a continuous electric-magnetic duality symmetry. In fact, the original BI theory as well as its gravitating generalization is symmetric under the $SO(2)$ electric-magnetic duality rotations. This duality can be extended to the $SL(2, R)$ symmetry when a dilaton and an axion are suitably added. However, the open string version of the BI action coupled to a dilaton does not enjoy electric-magnetic duality. This type of theory was discussed along similar lines in a recent paper [14] with results differing somewhat from those presented below. Meanwhile the $SL(2, Z)$ S -duality is now regarded as an exact symmetry of the superstring/M theory, so it is reasonable to concentrate on this version of the dilaton-coupled BI theory.

As was shown in [7, 12], the unique BI-dilaton-axion action generating the $SL(2, R)$ invariant equations of motion reads (with $G = \hbar = c = 1$)

$$\begin{aligned} S &= \frac{1}{4} \int \left\{ -R + 2(\nabla\phi)^2 + \frac{1}{2}e^{4\phi}(\nabla\kappa)^2 - \kappa F_{\mu\nu}\tilde{F}^{\mu\nu} + 4\beta^2(1 - \mathcal{R}) \right\} \sqrt{-g} d^4x \\ &= \int L \sqrt{-g} d^4x, \end{aligned} \quad (1.1)$$

where ϕ is a dilaton, κ is an axion and

$$\mathcal{R} = \sqrt{1 + \frac{F^2 e^{-2\phi}}{2\beta^2} - \frac{(F\tilde{F})^2 e^{-4\phi}}{16\beta^4}}, \quad (1.2)$$

with $F^2 = F_{\mu\nu}F^{\mu\nu}$, $F\tilde{F} = \tilde{F}_{\mu\nu}F^{\mu\nu}$. In the limit $\beta \rightarrow \infty$ this action reduces to the (truncated) heterotic string effective action in four dimensions.

In the BI theory, as in electrodynamics in media, one must distinguish between the field tensor $F_{\mu\nu}$ incorporating the electric strength and the magnetic induction, and the induction tensor $G_{\mu\nu}$ combining the electric induction and the magnetic field strength

$$G^{\mu\nu} = -\frac{1}{2} \frac{\partial L}{\partial F_{\mu\nu}} = \frac{e^{-2\phi}}{\mathcal{R}} \left(F^{\mu\nu} - \frac{(F\tilde{F})}{4\beta^2} \tilde{F}^{\mu\nu} e^{-2\phi} \right) + \kappa \tilde{F}^{\mu\nu}. \quad (1.3)$$

The Maxwell equations in terms of $G_{\mu\nu}$ are sourceless

$$dG = 0, \quad G = G_{\mu\nu} dx^\mu \wedge dx^\nu, \quad (1.4)$$

thus coinciding with the Bianchi identity for the field tensor

$$dF = 0, \quad F = F_{\mu\nu} dx^\mu \wedge dx^\nu. \quad (1.5)$$

Therefore the linear transformations

$$F \rightarrow aF + b\tilde{G}, \quad G \rightarrow cG - d\tilde{F}, \quad (1.6)$$

do not change this system of equations. Moreover, one can check that the equations for ϕ , κ following from the action (1.1) as well as the corresponding Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \quad (1.7)$$

$$T_{\mu\nu} = G_{\mu\lambda}F^\lambda{}_\nu + \phi_{,\mu}\phi_{,\nu} + \frac{1}{4}\kappa_{,\mu}\kappa_{,\nu}e^{4\phi} - g_{\mu\nu}L \quad (1.8)$$

remain also unchanged [12], provided the complex axidilaton $z = \kappa + ie^{-2\phi}$ undergoes the transformation

$$z \rightarrow \frac{cz - d}{bz + a} \quad (1.9)$$

with real a, b, c, d subject to $ac - bd = 1$. Note that these transformations change the action (1.1) by a total divergence.

In what follows we will consider either purely electric or purely magnetic configurations. In this case the axion decouples and can be consistently set to zero, with the term $F\tilde{F}$ in the lagrangian omitted. This truncated version of the theory still has a discrete electric-magnetic duality ($b = -d = 1, a = c = 0$):

$$F \rightarrow \tilde{G}, \quad G \rightarrow \tilde{F}, \quad \phi \rightarrow -\phi. \quad (1.10)$$

The plan of the paper is as follows. In Sec. 2 we perform the dimensional reduction of the action for static spherically symmetric purely electric (magnetic) configurations, and present the system in two alternative gauges. Sec. 3 is devoted to everywhere regular solutions which are studied both analytically and numerically. Black hole solutions are discussed in Sec. 4. Our results are summarized in Sec. 5. In the Appendix, the behavior of the solutions near the singularities is discussed in more detail. The numerical calculations presented in this paper were performed in collaboration with V.V. Dyadichev.

2 Basic equations

We start with the action, which is a truncated (for purely electric or magnetic configurations) version of (1.1)

$$S = \frac{1}{4} \int \left\{ -\frac{R}{G} + 2(\nabla\phi)^2 + 4\beta^2(1 - \mathcal{R}) \right\} \sqrt{-g} d^4x, \quad (2.1)$$

where we have restored the Newton constant G , and assume for \mathcal{R} a more general form with an arbitrary dilaton coupling constant γ :

$$\mathcal{R} = \sqrt{1 + \frac{F^2 e^{-2\gamma\phi}}{2\beta^2}}. \quad (2.2)$$

The discrete electric-magnetic duality (1.10) continues to hold for an arbitrary γ . This theory has altogether three dimensional parameters: G , β and γ with dimensionalities (in units $\hbar = c = 1$)

$$[G] = L^2, \quad [\beta] = L^{-2}, \quad [\gamma] = L, \quad (2.3)$$

We will be dealing with static spherically symmetric configurations, for which the equations of motion reduce to one-dimensional equations. Assuming the general static spherically symmetric parametrization of the metric

$$ds^2 = N\sigma^2 dt^2 - \frac{dr^2}{N} - R^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (2.4)$$

where N , R , σ are functions of r only, we perform a dimensional reduction leading to the Lagrangian

$$L = L_g + L_m \quad (2.5)$$

where the gravitational part is

$$L_g = \frac{1}{G} \left(N\sigma' R R' + \frac{\sigma}{2} ((NR)' R' + 1) \right), \quad (2.6)$$

while the matter part reads

$$L_m = -\frac{1}{2} \sigma N R^2 \phi'^2 + \sigma \beta^2 R^2 (1 - \mathcal{R}). \quad (2.7)$$

The dual induction two-form is reduced to

$$\tilde{G} = \tilde{G}_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{a' R^2 e^{-2\gamma\phi}}{\sigma \mathcal{R}} \sin \theta d\theta \wedge d\varphi \quad (2.8)$$

with

$$\mathcal{R} = \sqrt{1 - \left(\frac{a' e^{-\gamma\phi}}{\beta \sigma} \right)^2}. \quad (2.9)$$

Integrating the Maxwell equation $d\tilde{G} = 0$ one obtains

$$a' e^{-2\gamma\phi} = -\frac{QV\sigma}{R^2}, \quad \mathcal{R} = V \equiv \left(1 + \frac{Q^2 e^{2\gamma\phi}}{\beta^2 R^4} \right)^{-1/2}. \quad (2.10)$$

The dilaton equation reads

$$(\sigma N R^2 \phi')' = \sigma R^2 \gamma \beta^2 \left(\frac{1}{V} - V \right). \quad (2.11)$$

Variation of the action over R , N , σ gives the full set of Einstein equations

$$R(N\sigma')' + \frac{1}{2} [\sigma(NR)']' + \frac{1}{2} N(\sigma R')' = G \left(2\beta^2 \sigma R(1 - V) - \sigma N R \phi'^2 \right) \quad (2.12)$$

$$R(\sigma R')' - 2\sigma' R R' = -G \sigma R^2 \phi'^2 \quad (2.13)$$

$$(N R R')' - \frac{1}{2} (R'(NR)' + 1) = -G \left(\beta^2 R^2 \frac{(1 - V)}{V} + \frac{1}{2} N R^2 \phi'^2 \right). \quad (2.14)$$

Actually, the metric (2.4) is invariant under reparametrizations $r = r(\rho)$ of the radial coordinate, so that a gauge condition may be imposed. We will use two particular gauges in what follows: one is $R = r$, and another is $\sigma = 1$.

In the first gauge ($\underline{R = r}$),

$$ds^2 = N \sigma^2 dt^2 - \frac{dr^2}{N} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2.15)$$

Introducing a new variable $\psi = \beta^2 \gamma^4 Q^2 e^{2\gamma\phi}$, and performing rescalings

$$\beta \gamma r \rightarrow r, \quad \gamma \phi \rightarrow \phi, \quad (2.16)$$

(leading to a dimensionless independent variable r), the system of equations (2.11), (2.13), (2.14) may be rewritten as

$$Nr(r\phi')' + r\phi' = \frac{\psi}{\sqrt{\psi + r^4}} + 2gr\phi' \left(\sqrt{\psi + r^4} - r^2 \right) \quad (2.17)$$

$$(rN)' - 1 = g \left(2r^2 - 2\sqrt{\psi + r^4} - N r^2 \phi'^2 \right) \quad (2.18)$$

$$\sigma' = g \sigma r \phi'^2, \quad (2.19)$$

with the only dimensionless coupling constant

$$g = \frac{G}{\gamma^2}. \quad (2.20)$$

Another useful relation is the combination $2R(2.12) - 2N(2.13)$ which gives, after rescaling,

$$\left(\frac{(N\sigma^2)'r^2}{\sigma} \right)' = 4g\sigma r^2(1 - V). \quad (2.21)$$

In the second gauge ($\underline{\sigma = 1}$),

$$ds^2 = \lambda^2 dt^2 - \frac{d\rho^2}{\lambda^2} - R^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2.22)$$

the radial coordinate ρ being related to the radial coordinate r in the first gauge by $|d\rho/dr| = \sigma(r)$. We note that a suitable combination of Eqs. (2.11), (2.12) and (2.14) eliminates the source terms, leading to the equation

$$2(\lambda^2 R R')' - \frac{1}{2}(\lambda^2 R^2)'' + \frac{2G}{\gamma}(\lambda^2 R^2 \phi')' - 1 = 0. \quad (2.23)$$

The first integral of this equation, together with equations (2.12) and (2.11), leads (after rescaling ρ and R , as well as ϕ , as before), to the system

$$\frac{R'}{R} - \frac{\lambda'}{\lambda} + 2g\phi' = \frac{\rho - \rho_0}{\lambda^2 R^2}, \quad (2.24)$$

$$R'' = -gR\phi'^2, \quad (2.25)$$

$$(\lambda^2 R^2 \phi')' = u \equiv V \frac{\psi}{R^2}, \quad (2.26)$$

with ρ_0 an integration constant.

3 Regular solutions

In the gauge $R = r$ the conditions of asymptotic flatness read, in terms of rescaled quantities,

$$N = 1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right), \quad (3.1)$$

$$\phi = \phi_\infty + \frac{D}{r} + O\left(\frac{1}{r^2}\right), \quad (3.2)$$

$$\sigma = 1 - \frac{gD^2}{2r^2} + O\left(\frac{1}{r^3}\right). \quad (3.3)$$

Here M and D are the mass and the dilaton charge in the rescaled variables. Alternatively, one can introduce the unscaled quantities \overline{M} (via $N = 1 - 2G\overline{M}/r + \dots$) and \overline{D} , then $M = G\beta\gamma\overline{M}$, $D = \beta\overline{D}$.

For these quantities one can derive useful sum rules. Define a local mass function $m(r)$ through

$$N = 1 - \frac{2m(r)}{r}, \quad (3.4)$$

with $m(r) \rightarrow M + O(1/r)$ as $r \rightarrow \infty$. Integrating Eq. (2.21) from some finite r_1 to infinity and taking into account the conditions of asymptotic flatness one obtains

$$M = 2g \int_{r_1}^{\infty} \sigma r^2 (1 - V) dr + \frac{(N\sigma^2)'r^2}{2\sigma} \Big|_{r=r_1}. \quad (3.5)$$

For solutions regular at the origin the last term vanishes as $r_1 \rightarrow 0$, so one obtains

$$M = 2g \int_0^{\infty} \sigma r^2 (1 - V) dr. \quad (3.6)$$

In a similar way one can derive from the dilaton equation

$$D = \int_{r_1}^{\infty} \sigma r^2 (V - V^{-1}) dr - \sigma N r \xi \Big|_{r=r_1}, \quad (3.7)$$

with $\xi = r\phi'$. Thus for the regular solutions we have

$$D = \int_0^{\infty} \sigma r^2 (V - V^{-1}) dr. \quad (3.8)$$

Combining these two formulas we find the relation

$$M + gD = -g \int_0^{\infty} \sigma r^2 \frac{(1 - V)^2}{V} dr, \quad (3.9)$$

showing that the dilaton charge for the electric solution is negative ($V < 1$) and that its absolute value is greater than the mass.

To analyse the behavior of regular solutions near the origin it is convenient to introduce the logarithmic variable $\tau = \ln r$. The system (2.17)–(2.19) may be rewritten as the system of first order equations

$$\dot{\psi} = 2\xi\psi, \quad (3.10)$$

$$N\dot{\xi} = u - \xi v, \quad (3.11)$$

$$\dot{N} = v - N(1 + g\xi^2), \quad (3.12)$$

$$\dot{\sigma} = g\sigma\xi^2, \quad (3.13)$$

where

$$u = \frac{\psi}{\sqrt{\psi + r^4}}, \quad v = 1 + 2g \left(r^2 - \sqrt{\psi + r^4} \right). \quad (3.14)$$

By definition, the regular solution should be flat in the vicinity of the origin $r = 0$ ($\tau \rightarrow -\infty$), i.e. $N = 1$ at the origin. Under this condition one can find the series solution in powers of $1/\tau$ with logarithmic coefficients

$$\psi = \frac{1}{\tau^2} \left(1 + \frac{2L}{\tau} + \frac{3L^2 - 2(2g - 1)L + 4(1 - g)}{\tau^2} + \dots \right), \quad (3.15)$$

$$\xi = \frac{1}{\tau} \left(-1 + \frac{-L + 2g - 1}{\tau} - \frac{L^2 - 3(2g - 1)L + 4(1 - g)^2 + 1}{\tau^2} + \dots \right), \quad (3.16)$$

$$N = 1 + \frac{2g}{\tau} \left(1 + \frac{L + 1/2}{\tau} + \frac{L^2 + 2(1 - g)(L + 3/2)}{\tau^2} + \dots \right). \quad (3.17)$$

In these formulas

$$L = (2g - 1) \ln(-\tau) + c, \quad (3.18)$$

where c is a free parameter. One can also show from Eq. (2.21) that $(N\sigma^2) \simeq (4g/3)\sigma e^{2\tau}$ is vanishingly small, so that $\lambda^2 = N\sigma^2$ is constant to all orders in this expansion. Two particular values of g should be mentioned. The first one is $g = 0$ which corresponds to a vanishing Newton constant. In this case the expansion (3.17) generates $N \equiv 1$, so we get the flat space dilaton-BI solution. The second is $g = 1/2$, in this case there are no logarithmic terms in the expansions.

These local solutions being continued (numerically) to large r meet the conditions of asymptotic flatness for continuously varying c subject to the condition

$$c > c_{cr}(g). \quad (3.19)$$

The dependence of c_{cr} on the effective coupling constant g is shown in Fig.1. The limit $g = 0$ corresponds to the flat space solutions, in this case $N \equiv 1$. The corresponding value is approximately $c_{cr} = 1.52$. When g is large enough the critical value moves to the negative half-plane. The variation with c_{cr} of the mass, dilaton charge and the asymptotic value of ψ are shown in Fig. 2. It is worth emphasizing that we deal with a one-parameter family, i.e. for a given mass the values of the dilaton charge and the electric charge (proportional to the square root of ψ_∞) are fixed. Physically these particle-like solutions can be regarded as the original Born-Infeld field distributions of the point charge deformed by gravity and by the dilaton. Thus the mere existence of regular particle-like solutions in the present theory is by no means surprising.

When the parameter c decreases and approaches the critical value, both the mass and the absolute value of the dilaton charge increase, with an approximate power law dependence $M \propto (c - c_{cr})^p$, where $p = 1/(1 + g)$ (see Fig. 2), and at the same time

$$g|D| \rightarrow M \quad \text{as} \quad c \rightarrow c_{cr}. \quad (3.20)$$

This can be understood qualitatively using the sum rules above. When c is close to its boundary value the main contribution to the integral comes from the region of large r where $V \sim 1$. Then from Eqs. (3.6), (3.8), (3.9) it is clear that if the integral for M diverges for $r \rightarrow \infty$, the integral for the sum $M + gD$ is less divergent than both M and $|D|$. The numerical results also show that the ratio $\psi(\infty)/MD$ goes to a finite limit as $c \rightarrow c_{cr}$. To understand this, note that a generic asymptotically flat solution of the non linear theory is asymptotic for $r \rightarrow \infty$ ($V \rightarrow 1$) to an electric solution of the linear Einstein-Maxwell-dilaton theory (obtained from the magnetic solution of [11] by transforming $\phi \rightarrow -\phi$ and rescaling ρ , R and ϕ)

$$\begin{aligned} \psi &= \frac{\rho_+ \rho_-}{g+1} \left(1 - \frac{\rho_-}{\rho}\right)^{2/(g+1)}, \quad R = \rho \left(1 - \frac{\rho_-}{\rho}\right)^{1/(g+1)}, \\ \lambda^2 &= \left(1 - \frac{\rho_+}{\rho}\right) \left(1 - \frac{\rho_-}{\rho}\right)^{(g-1)/(g+1)} \end{aligned} \quad (3.21)$$

($g = \gamma^{-2}$ for $G = 1$). Comparing with (3.1), we see that $M + gD = (\rho_+ - \rho_-)/2$, so that the limit $M + gD \ll M$ corresponds to the BPS limit $\rho_- \rightarrow \rho_+$. In this limit, we obtain from (3.21)

$$\psi(\infty) \simeq (g+1) \left(\frac{M}{g}\right)^2. \quad (3.22)$$

This relation is well confirmed by the numerical observations (Fig. 2). Relations (3.22) and (3.20) together imply

$$M^2 + g^2 D^2 = Q_{eff}^2, \quad Q_{eff} = g \left(\frac{2}{g+1} \psi(\infty) \right)^{1/2}, \quad (3.23)$$

which corresponds to the BPS saturation of the Einstein-Maxwell-dilaton theory [9].

The transition point in the parameter space $c = c_{cr}$ corresponds to a limiting solution which is not asymptotically flat. To find its asymptotic behaviour, consider the asymptotic solution (3.21) in the BPS limit $\rho_- \rightarrow \rho_+$. The limit in which both ρ and $M = g\rho_+/(g+1)$ go to infinity corresponds to the near horizon limit $|\rho - \rho_+| \ll \rho_+$. Putting $(\rho - \rho_+)/\rho_+ = (r/\rho_+)^{g+1}$, rescaling t , and taking the limit $\rho_+ \rightarrow \infty$, we arrive at the asymptotic critical solution

$$ds_{cr}^2 = a^2 r^{2g} dt^2 - (g+1)^2 dr^2 - r^2 d\Omega^2, \quad \psi_{cr} = r^2/(g+1). \quad (3.24)$$

This asymptotic behaviour has not been directly tested, as in all our numerical computations the parameter c was at best only approximately equal to its critical value. Nevertheless the critical solution can be approached numerically as the envelope of asymptotically flat regular solutions with $c \simeq c_{cr}$. In this respect, we note that the curves for $\xi(\tau)$ show increasingly large and flat maxima near $\xi = 1$ (before dropping again to 0 as $\tau \rightarrow \infty$), in agreement with the asymptotic critical behaviour $\xi_{cr}(\infty) = 1$ from (3.24).

For $c < c_{cr}$ the solution starting at the origin as (3.15,3.16,3.17) develops a coordinate singularity at some radius $r = r^*$. In the gauge $R = r$ one observes that for $r \rightarrow r^*$ the metric function N approaches zero while the dilaton has a square root singularity

$$N \sim (r^* - r), \quad \phi \sim \sqrt{r^* - r}, \quad \xi \sim \frac{1}{\sqrt{r^* - r}}. \quad (3.25)$$

Such a behaviour can be derived by defining locally a new radial coordinate x :

$$r = r^* \exp(-gx^2/2), \quad (3.26)$$

and assuming $\psi \simeq \psi^*(1 - 2bx)$, where b is some constant. Then $\xi \simeq b/gx$, and Eq. (3.11) yields $N \simeq -gv^*x^2$, with $v^* < 0$ ($gv^* > r^{*2} + 1/4g$) by continuity, while Eq. (3.12) yields $b^2 = 1$, so that $b = \pm 1$. The complete local solution

$$\begin{aligned} \xi &= \pm(1/gx) + d + \dots, \\ N &= gx^2(-v^* \mp g(u^* + dv^*)x + \dots), \\ \sigma &= k(1/x \mp 2gd + \dots) \end{aligned} \quad (3.27)$$

depends on the four integration constants r^* , ψ^* , d and k , and so is generic.

Obviously the coordinate patch $r < r^*$ fails to describe the full solution. The extension may be achieved by going to the second gauge (2.22), as from (3.26) and the last equation (3.27) $d\rho \propto dx$ near $R \equiv r = r^*$. From Eq. (2.26), $R^2\lambda^2\phi'$ is an increasing function of ρ . So if one starts from a solution regular near $R = 0$ ($\phi'_\rho \propto \phi'_r > 0$) and increases R and ρ , one must have $\phi'_\rho > 0$ at $R = r^*$, with $R'_\rho = 0$. This is a maximum of R from Eq. (2.25). As ρ is further increased, R must decrease to zero, while ϕ'_ρ stays positive so that ϕ continues to increase, and a regular solution at $R = 0$ ($\phi \rightarrow -\infty$) cannot be

achieved. The resulting solution is compact and singular ('bag of gold'), with the power law behaviours near the singularity

$$\psi \propto r^{2\delta}, \quad N \propto r^{-1-g\delta^2}, \quad \sigma \propto r^{g\delta^2} \quad (g > 1/4), \quad (3.28)$$

$$\psi \propto r^{-1/g}, \quad N \propto r^{-1/2g}, \quad \sigma \propto r^{1/4g} \quad (g < 1/4), \quad (3.29)$$

where $\delta < 0$ is bounded by

$$\sqrt{4 + 3g^{-1}} - 2 < -\delta < \sqrt{g^{-1}}. \quad (3.30)$$

(details on the derivation of these behaviours and bounds are given in the Appendix).

Let us consider now magnetically charged solitons. Reverting to unscaled variables and introducing a magnetic charge via

$$F = P \sin \theta d\theta \wedge d\varphi \quad (3.31)$$

we obtain

$$\mathcal{R} = \sqrt{1 + \frac{P^2 e^{-2\gamma\phi}}{\beta^2 R^2}} \quad (3.32)$$

Rescaling the variables in the gauge $R = r$ we obtain the same system (2.17)–(2.19) with $\xi \rightarrow -\xi$, $Q \rightarrow P$. This is in a perfect agreement with the electric-magnetic duality discussed in the Introduction. Accordingly, the electric solitons described in this section can be reinterpreted as magnetic solitons. In this case the parameter δ in (3.28) is positive, and it is easy to see that the bounds above still hold if $-\delta$ is replaced by $|\delta|$. It is worth noting that the above symmetry holds in the Einstein frame, in the conformally related 'string' frame the metrics of electric and magnetic solutions are essentially different.

4 Black holes

To study black holes, let us assume the existence of a non-degenerate horizon at some $r = r_h$, i.e. $N(r_h) = 0$, with $N'(r_h) > 0$. It is again convenient to work with the rescaled first order differential system (3.10-3.13) with the logarithmic variable $\tau = \ln(r/r_h)$. From this system we find the following series solution

$$N = v_h \tau + \frac{1}{2v_h} (v_1 v_h - v_h^2 - g u_h^2) \tau^2 + O(\tau^3), \quad (4.1)$$

$$\xi = \frac{u_h}{v_h} + \frac{1}{2v_h^2} (v_h u_1 - v_1 u_h) \tau + O(\tau^2), \quad (4.2)$$

$$\psi = b + \frac{2b u_h}{v_h} \tau + O(\tau^2), \quad (4.3)$$

Here $u_h = u(r_h)$,

$$v_1 = 2g \left(2r_h^2 - \frac{b u_h + 2v_h r_h^4}{v_h \sqrt{b + r_h^4}} \right), \quad u_1 = \frac{b}{(b + r_h^4)^{3/2}} \left(2r_h^4 \left(\frac{u_h}{v_h} - 1 \right) + b \frac{u_h}{v_h} \right), \quad (4.4)$$

and $b = \psi(r_h)$ is a free parameter varying in the finite interval

$$-r_h^4 < b < \frac{1}{g} \left(\frac{1}{4g} + r_h^2 \right). \quad (4.5)$$

The left bound corresponds to the condition of positivity under the square root in V , while the right bound comes from the assumption $v_h > 0$.

Actually the right bound on the parameter b for asymptotically flat solutions is lower, namely

$$b < b_{cr}(g) < \frac{1}{g} \left(\frac{1}{4g} + r_h^2 \right), \quad (4.6)$$

otherwise the bag of gold type singularity is met. The critical b depends on the horizon radius as well. On Fig. 4 the curves $b_{cr}(g)$ are shown for various values of r_h . For small r_h the exterior black hole solutions approach regular solutions. As in the case of regular solutions with c close to c_{cr} , both the mass and the dilaton charge grow with b . From the Eqs. (3.5) and (3.7) one can see that the boundary term in the black hole case vanishes for the dilaton but remains finite for the mass:

$$M = M_0 + 2g \int_{r_h}^{\infty} \sigma r^2 (1 - V) dr, \quad M_0 = \frac{v_h \sigma_h r_h^2}{2} \quad (4.7)$$

Here the first term may be regarded as the 'bare' mass of the black hole, and the second as a contribution from the black hole hair. The sum rule for the dilaton charge preserves its form (3.8), where the integration now is performed over the exterior space:

$$D = \int_{r_h}^{\infty} \sigma r^2 (V - V^{-1}) dr \quad (4.8)$$

The combined sum rule now reads

$$M - M_0 + gD = -g \int_{r_h}^{\infty} \sigma r^2 \frac{(1 - V)^2}{V} dr. \quad (4.9)$$

When b_{cr} is approached this quantity remains finite while M and D diverge, so that asymptotically $g|D|$ tends to the field mass $M - M_0$.

From Eq. (2.25), $R(\rho)$ must vanish for some $\rho_s < \rho_h$, leading to a singularity. There can be no inner Cauchy horizon, so that this singularity must be timelike. To show this, note that from Eq. (2.26), $\lambda^2 R^2 \phi'$ must decrease when ρ decreases inside the event horizon $\rho = \rho_h$, and thus must stay negative (ϕ'_h is positive, while $\lambda^2 < 0$ for $r < r_h$), so that λ^2 cannot vanish again. Reversing the sign of τ in the Eqs. (3.10)–(3.13) one can integrate inside the black hole up to the singularity. The leading terms near the singularity are the same as in Eq. (3.28), with now δ positive for electric black holes and negative for magnetic black holes,

$$m \simeq \frac{\mu}{r g \delta^2}, \quad \xi \simeq \delta, \quad \psi \simeq \nu^2 r^{2\delta}, \quad \sigma \simeq a r^{g\delta^2} \quad (4.10)$$

where $\delta, \mu > 0$, $\nu > 0$ and $a > 0$ are free parameters. The solution (4.10) is generic, so starting from the expansions (4.1)–(4.3) inside the horizon one always meets a member of the local family (4.10) with certain μ, ν, a, δ .

The parameter δ is constrained by various bounds (see the Appendix). One of these is obtained from the combined sum rule similar to (4.9), but with the integration covering the internal space,

$$\mu a (1 - g\delta^2 + 2g|\delta|) - M_0 = g \int_0^{\rho_h} R^2 \frac{(1 - V)^2}{V} d\rho. \quad (4.11)$$

On account of the positivity of M_0 and μa , this implies $1 - g\delta^2 + 2g|\delta| > 0$, i.e.

$$|\delta| < 1 + \sqrt{1 + g^{-1}}. \quad (4.12)$$

Another bound, obtained by combining the sum rule (4.11) with the first integral (2.24) evaluated at the singularity, is

$$g|\delta| < \rho_h/2\mu a - 1. \quad (4.13)$$

In the limit $g \rightarrow \infty$, corresponding to $\gamma \rightarrow 0$, the dilaton decouples, and the bound (4.13) leads, for fixed (ρ_h, μ, a) (note that the ratio $\rho_h/\mu a$ is invariant under the rescaling (2.16)), to $\delta = O(g^{-1})$, so that the exponents in (4.10) go to zero, and the mass function goes to a constant at the singularity $r = 0$, consistently with the results of [7, 8] for Einstein-Born-Infeld black holes¹. Thus the dilaton reinforces a divergence of the mass function at the singularity.

5 Conclusion

Let us summarize our results. We have shown that the Einstein-Born-Infeld-dilaton theory admits a one-parameter family of particle-like globally regular solutions characterized by their mass, electric (magnetic) charge, and dilaton charge. The unique parameter determining these three quantities is bounded from below, and when the boundary is approached, the charges exhibit a BPS saturation similar to that of the BPS black holes of Einstein-Maxwell-dilaton theory (recall that the latter does not possess particle-like solutions). In this limit the absolute values of charges rise indefinitely, as well as the effective radius of the particle, so that the main contribution to the total charges comes from large radii. Since at large radii the Born-Infeld non-linearity is small, the correspondence with the Maxwellian counterpart of the theory is by no means surprising. However the BPS saturation property of regular solutions at large masses is not obvious *a priori*.

Another type of solutions regular at the origin are compact, with a point singularity at the ‘other end’ (such configurations are commonly called ‘bags of gold c’). For these solutions, the areas of two-sphere sections increase up to some finite limiting value and then decrease again up to the final curvature singularity. These solutions also form a one-parameter sequence, corresponding to values of the parameter below the threshold for particle-like solutions.

We have also found a two-parameter family of black-hole solutions for arbitrary values of the horizon radius and the second free parameter varying on a finite interval. Alternatively, one can consider as independent parameters the mass and the electric (magnetic) charge of the black hole. The dilaton charge is a dependent quantity like in the linear Einstein-Maxwell-dilaton theory. Altogether these three quantities satisfy a BPS inequality which is saturated in the limit of the infinite mass and the dilaton charge. The black holes do not possess internal Cauchy horizons and the singularity inside them is always spacelike. This has to be contrasted with the Born-Infeld black holes without dilaton in which case the solutions possessing an internal Cauchy horizon still exist. Like in the linear Maxwell-dilaton case, the scalar field prevents the formation of an internal horizon. The local mass function has a power-low behaviour near the singularity with a power

¹Note that (4.11) implies $\mu > 0$ in the limit $g \rightarrow \infty$. Here there is only one horizon for finite g , so the two-horizon case ($\mu < 0$) cannot be obtained as a limit.

index depending on the global black hole parameters. Solutions with a regular event horizon exist also in the bag of gold (compact) form, these are doubly singular.

The above picture holds in the Einstein frame. One should inquire how the regular solutions look in the 'string' frame which is related to the present one by a conformal transformation with the conformal factor ψ . According to (3.15), this conformal factor vanishes logarithmically as $r \rightarrow 0$ for electric solutions (and diverges for magnetic ones), so the 'string' metric in the vicinity of the origin is singular. Nevertheless, the ADM mass remains finite in this frame both for electric and magnetic solutions.

It is worth comparing our results with those of the paper [14] where another type of coupling of the Born-Infeld theory to dilaton was assumed. In the latter case there is no symmetry between electric and magnetic solutions, in particular, the magnetic non-singular particle-like solutions were found in the string frame (contrary to electric ones). Our version of the theory is (classically) S -dual and, moreover, it inherits the structure of the $\mathcal{N} = 4, D = 4$ supergravity. Therefore in the Einstein frame magnetic solutions are related to electric ones simply by reversing the sign of the dilaton. The string metric is singular at the origin in both cases.

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Appendix: Power law behaviours near the singularity

Let us first consider the electric bag of gold of Sect. 3 with a regular origin and a point singularity. Assume the power law behaviours

$$\psi \simeq \nu^2 r^{2\delta}, \quad \sigma \simeq a r^{g\delta^2} \quad (\text{A.1})$$

(with the integration constants $\delta < 0$, $\nu > 0$, $a > 0$). Then, if

$$g\delta^2 + \delta + 1 > 0, \quad (\text{A.2})$$

Eqs. (2.17)-(2.18) are solved near the singularity $r = 0$ by

$$N \simeq -2\mu r^{-1-g\delta^2}, \quad (\text{A.3})$$

with $\mu < 0$. Several bounds on the exponent δ can be derived. First, Eq. (2.24) where $\rho_0 = -(\lambda^2 R/\sigma)(1 + 2g\xi)|_{r=0} = 0$ for regular solutions gives, near the singularity $\rho = \rho_1$,

$$g\delta^2 - 4g\delta - 3 = -\rho_1/\mu a > 0, \quad (\text{A.4})$$

leading to the bound

$$-\delta > \sqrt{4 + 3g^{-1}} - 2. \quad (\text{A.5})$$

Other bounds can be obtained from sum rules obtained in a manner similar to (3.6), (3.8), (3.9). These sum rules are

$$\mu a(g\delta^2 - 1) = 2g \int_0^{\rho_1} R^2(1 - V)d\rho, \quad (\text{A.6})$$

$$2\mu a\delta = \int_0^{\rho_1} R^2(V^{-1} - V)d\rho, \quad (\text{A.7})$$

$$\mu a(1 - g\delta^2 + 2g\delta) = g \int_0^{\rho_1} R^2 \frac{(1 - V)^2}{V} d\rho, \quad (\text{A.8})$$

where all the right-hand sides are finite and positive. From Eqs. (A.6) and (A.8) we obtain the bounds

$$\sqrt{1 + g^{-1}} - 1 < -\delta < \sqrt{g^{-1}}, \quad (\text{A.9})$$

while Eq. (A.7) gives nothing new. It is easy to show that the lower bound (A.9) is weaker than the bound (A.5), so that our final bounds are

$$\sqrt{4 + 3g^{-1}} - 2 < -\delta < \sqrt{g^{-1}}. \quad (\text{A.10})$$

A corollary of Eq. (A.10) is that the singular behaviour (3.28) can exist only for

$$g > 1/4, \quad (\text{A.11})$$

which ensures the inequality (A.2). Also, the combination of (A.4) and (A.8) leads to the inequality

$$-g\delta < 1 - \rho_1/2\mu a. \quad (\text{A.12})$$

Assume now the power law behaviours (A.1) with

$$g\delta^2 + \delta + 1 < 0, \quad (\text{A.13})$$

which is possible only for

$$g < 1/4. \quad (\text{A.14})$$

Then the behaviour

$$N \simeq \alpha r^\delta \quad (\text{A.15})$$

solves Eq. (2.18) if $\alpha = -2g\nu/(1 + \delta + g\delta^2) > 0$, and Eq. (2.17) if $1 + 2g\delta = 0$, leading to the behaviours (3.29). Eq. (2.24) does not give new information, neither do the sum rules analogous to (A.7) and (A.8) whose both sides diverge with the same power law and sign, while the sum rule analogous to (A.6) shows that the behaviour of λ^2 near the singularity must be $\lambda^2 \simeq \alpha a^2 - \beta(\rho_1 - \rho)^\gamma$, with $\gamma = (1 - 4g)/(1 + 4g) > 0$, $\beta > 0$. For $g = 1/4$, (A.15) is replaced by $N \simeq -(\nu/2)r^{-2} \ln r$.

Consider now the black hole case. Let us assume again the behaviours (A.1), with $\delta > 0$ for electric black holes. In this case $g\delta^2 + |\delta| + 1 > 0$ for all $g \geq 0$, so that the behaviour of N is always given by (A.3) with $\mu > 0$. As in the case of the bag of gold singularity, we can derive various bounds on $|\delta|$. On the horizon Eq. (2.24) gives

$$\rho_0 - \rho_h = r_h^2(\lambda^2)'(\rho_h)/2 = M_0 > 0. \quad (\text{A.16})$$

Applying now the same equation (2.24) near the singularity $\rho \propto r^{g\delta^2+1} \rightarrow 0$ and inserting the preceding result, we obtain to leading order

$$g\delta^2 - 4g\delta - 3 = -\frac{(\rho_h + M_0)}{\mu a}. \quad (\text{A.17})$$

The negativity of the right-hand side leads to the bound

$$\delta < 2 + \sqrt{4 + 3g^{-1}}. \quad (\text{A.18})$$

On the other hand, the sum rules obtained as in the bag of gold case give

$$\mu a(g\delta^2 - 1) + M_0 = 2g \int_0^{\rho_h} R^2(1 - V)d\rho, \quad (\text{A.19})$$

$$2\mu a\delta = \int_0^{\rho_h} R^2(V^{-1} - V)d\rho, \quad (\text{A.20})$$

$$\mu a(1 - g\delta^2 + 2g\delta) - M_0 = g \int_0^{\rho_h} R^2 \frac{(1 - V)^2}{V} d\rho. \quad (\text{A.21})$$

This last sum rule (with M_0 and μa positive) leads to the bound

$$\delta < 1 + \sqrt{1 + g^{-1}}, \quad (\text{A.22})$$

which is stronger than the bound (A.18). Again, the combination of (A.17) and (A.21) leads to the inequality

$$g\delta < \rho_h/2\mu a - 1. \quad (\text{A.23})$$

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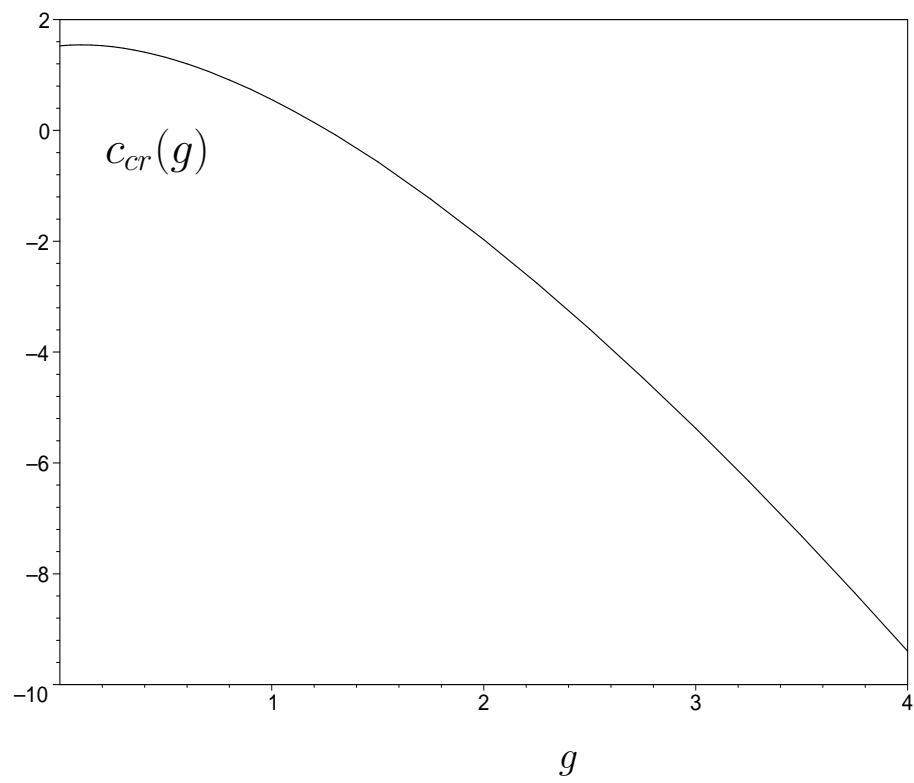


Figure 1: Critical parameter c_{cr} as a function of the dimensionless coupling constant g .

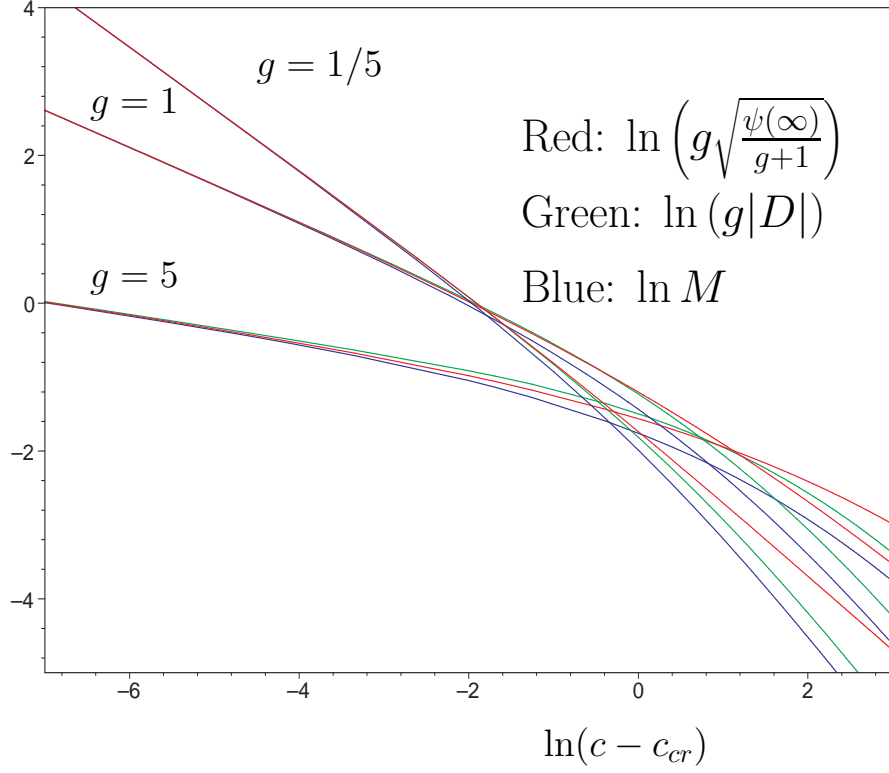


Figure 2: Dependence of the mass and the suitably normalized dilaton charge and $\psi(\infty)$ on c . When c approaches the critical value all these quantities converge to the same value $M = g|D| = g(\psi(\infty))^{1/2}(g+1)^{-1/2}$. This corresponds to the BPS saturation in the linear theory.

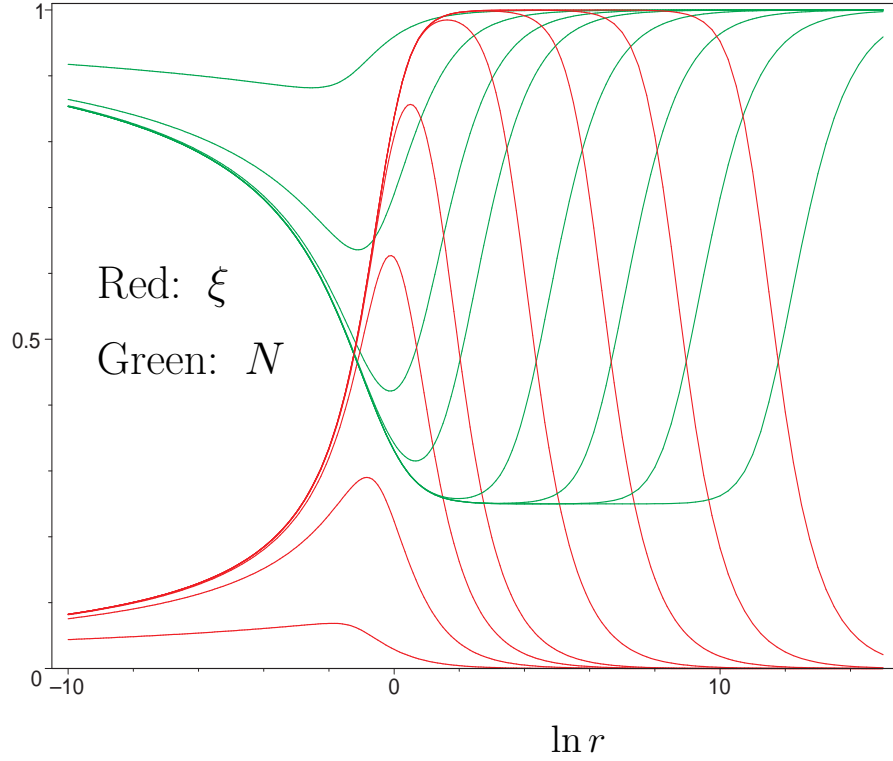


Figure 3: A family of regular solutions with $g = 1$ for different values of $c - c_{cr}$ approaching zero (to the right). One can see that at the intermediate r -region both curves $N(r)$, $\xi(r)$ approach the limiting (asymptotically non-flat) solution $\xi \sim 1$, $N \sim 0.25$: this region expands to the right when $c \rightarrow c_{cr}$. Thus the effective radius of the Einstein-Born-Infeld-dilaton particles increases when the boundary $c = c_{cr}$ is approached.

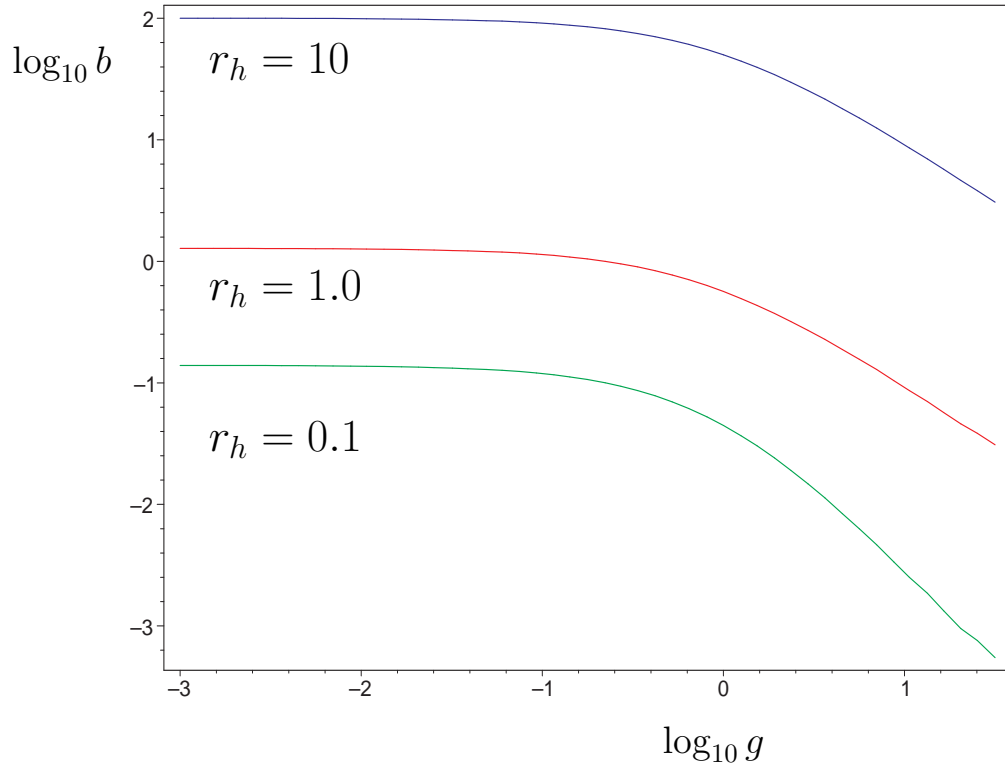


Figure 4: Critical parameter $b = \psi(r_h)$ as a function of the coupling constant for different radii of the horizon.

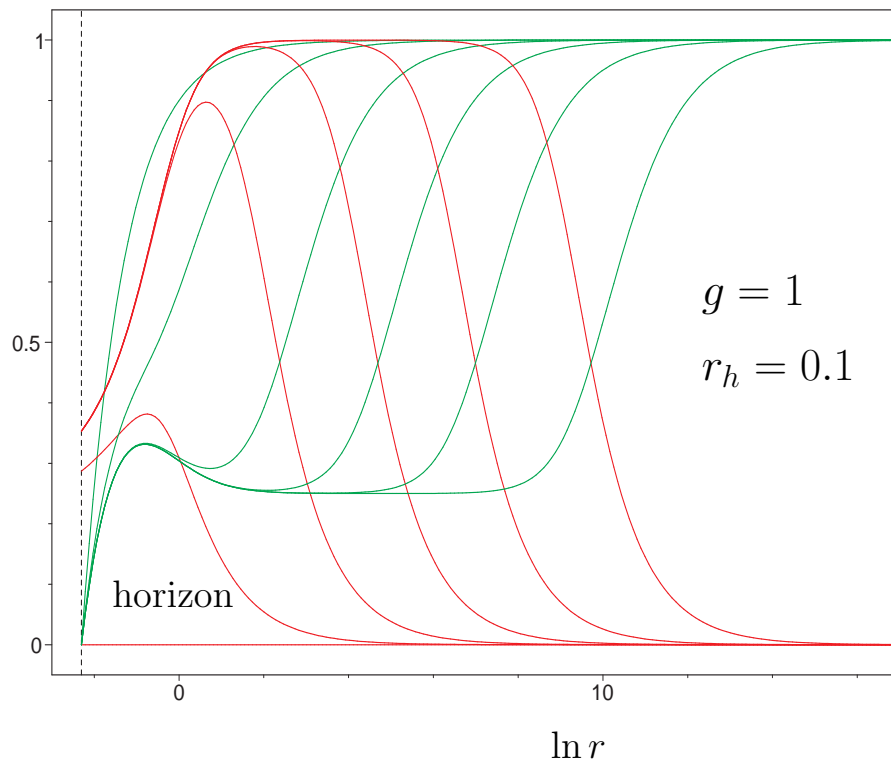


Figure 5: A family of black hole solutions with $g = 1$ for $b_{cr} - b$ approaching zero (to the right) for 'small' black hole $r_h = 0.1$. One observes features similar to the regular case. At the left side the behavior is modified due to the presence of the event horizon.

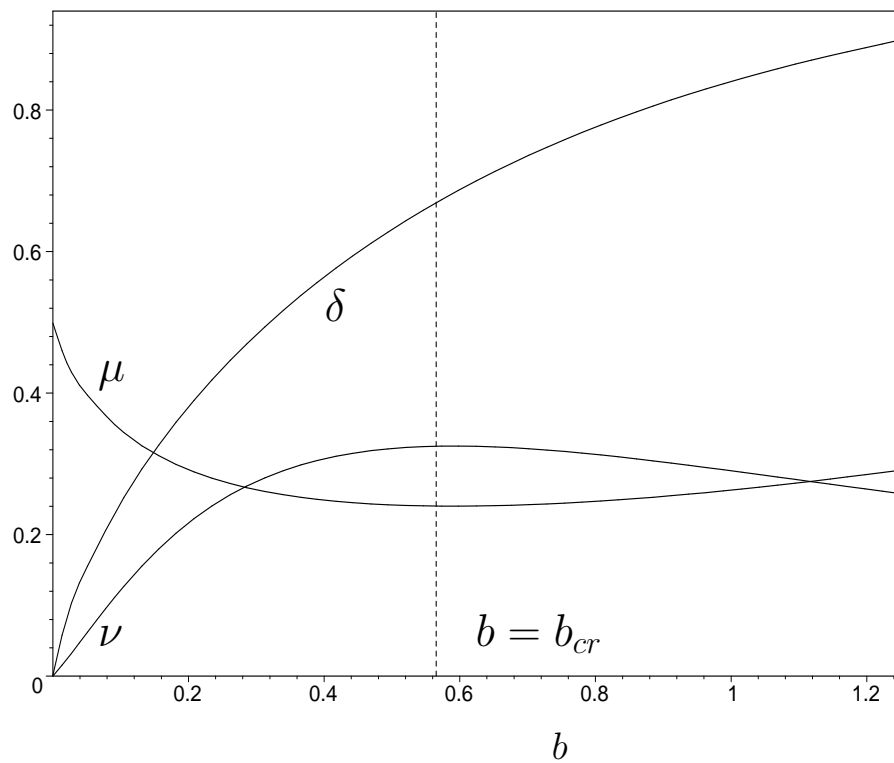


Figure 6: Parameters of the local solution near the singularity versus b for $g = 1$, $r_h = 1$. Only the region $b < b_{cr}$ corresponds to asymptotically flat solutions (black holes). For $b \rightarrow 0$ one recovers the Schwarzschild solution.